ALM - Stochastic Term Structure Models

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e## Course Program

- Basic interest rate theory
- Interest rate risk management
- `Stochastic term structure models`
- Risk measurement
- Reinsurance and insurance-linked securities
- Mean-variance analysis for ALM

Contents of the chapter

- Stochastic term structure models (a.k.a. `short rate` models).
    - Cox-Ingersoll-Ross model
    - Vasicek model
    - Hull-White model
    - Ho-Lee model
- Simulation of stochastic evolution of the yield curve.

Introduction 1

- We have seen that
  - `Matching` protects the surplus against all changers in yield curve, but is usually not useful in practice.
  - `Immunization` protects the surplus agains parallel shifts in the yield curve, but not necessarily against other kinds of movement.
- To be able to assess what changes one can *reasonably* expect, one must model the stochastic evolution of the yield curve - the `term structure`.

Introduction 2

- The purpose of `stochastic term structure models` is to represent the evolution of the yield curve as a stochastic process.
  - At time t, the market price of €1 payable at time $T > t$ is $P(t, T)$. This price varies `randomly` from day to day.
  - On any given day t, the function $\{P(t, T) : T > t\}$ can be determined, at least in principle, from the observed market prices of bonds and bills.
- We write $P(t, T) = exp(-R(t, T)(T - t))$, so that $R(t, T)$ is the random spot rate (`zero rate`) at time t for a term of $T - t$.

Introduction 3

- The `short rate` is the interest rate at which one can borrow money for an infinitesimal period of time from time t:

$$r(t) \stackrel{\mathrm{def}}{=} \lim_{T \to t} R(t, T).$$

- If we write

$$P(t, T) = E\left(e^{-\int_t^T r(s)ds}\right)$$

Introduction 4

- Then

$$R(t,T) = -\frac{1}{T-t} ln\ E\left(e^{-\int_t^T r(s)ds}\right)$$

Term structure equilibrium models

- A `stochastic model` of the evolution of the short rate $r(t)$ together with a no-arbitrage assumption implies the entire yield curve at time t.
  - `Vasicek` model

$$dr(t) = a(b - r(t))dt + \sigma dz(t)$$

  - Cox-Ingersoll-Ross model - `CIR`

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dz(t)$$

- The term $z(t)$ is a standard Brownian motion (white noise).

Term structure equilibrium models

- Both the Vasicek and CIR model have `mean reversion`.
- The parameter b describes the long-term average rate.
- The parameter a describes the strength of the gravitation back to the average rate.
- The Vasicek model `allows negative` interest rates.
- The CIR model `allows only positive` interest rates.

Affine term structure

- The term structure of the Vasicek and CIR model has the affine form

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

- Equivalently,

$$R(t,T) = \frac{1}{T-t}(-ln\ A(t,T) + B(t,T)r(t))$$

with functions A and B that depend on a, b and σ.

Term structure - Vasicek model

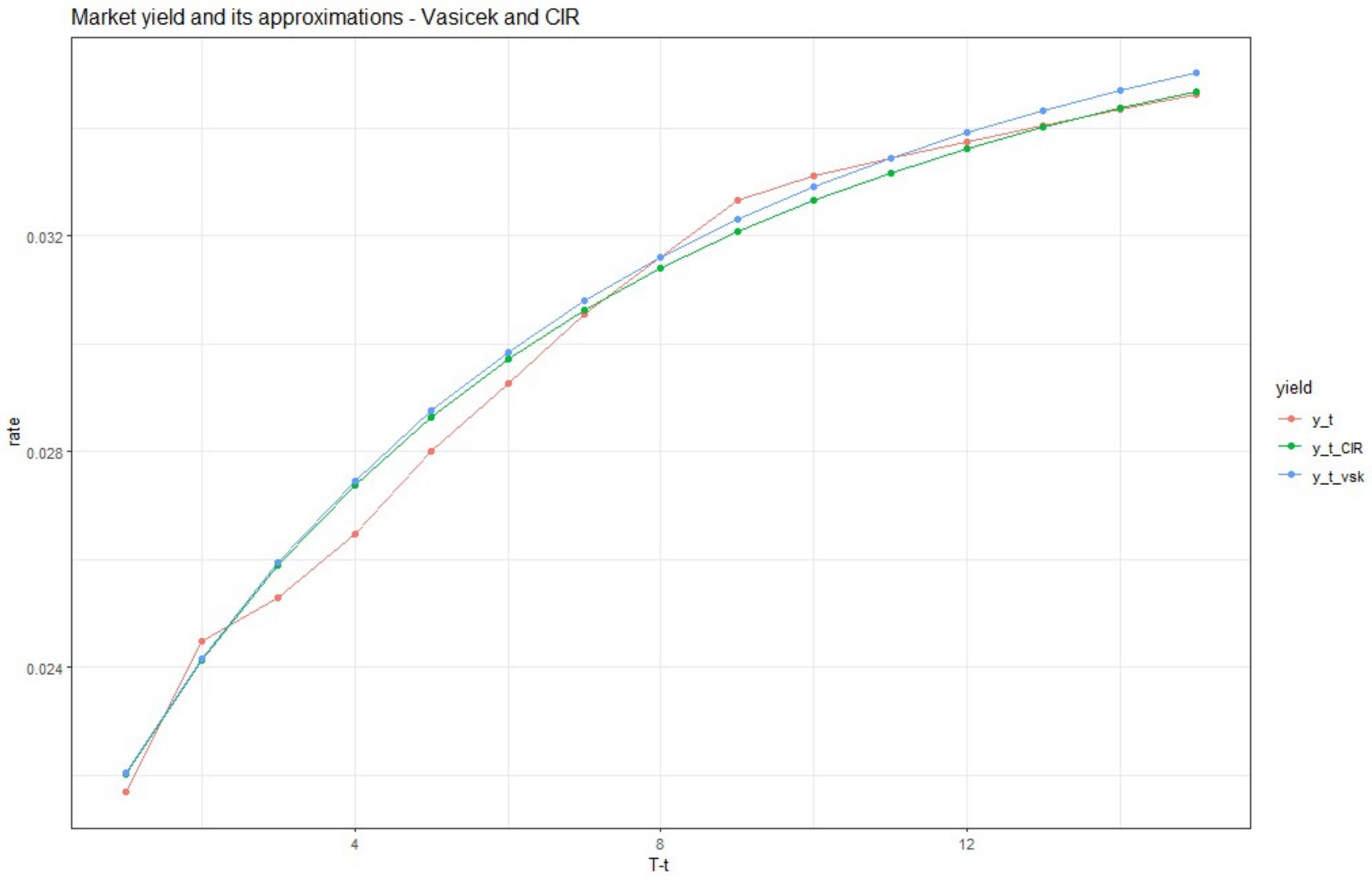
- The term structure is easy to calculate and simulate once the parameters are given or estimated.
- $P(t,T) = A(t,T)e^{-B(t,T)r(t)}$
- $B(t,T) = \frac{1-e^{-a(t-T)}}{a}$
- $A(t,T) = e^{\frac{(B(t,T)-(T-t))(a^2b-\frac{1}{2}\sigma^2)}{a^2}} \cdot e^{-\frac{\sigma^2 B^2(t,T)}{4a}}$

Term structure - CIR model

- The term structure is easy to calculate and simulate once the parameters are given or estimated.
- $P(t,T) = A(t,T)e^{-B(t,T)r(t)}$
- $A(t,T) = \left(\frac{2\gamma e^{(\gamma+a)(T-t)/2)}}{(\gamma+a)(e^{\gamma(T-t)}-1)+2\gamma}\right)^{2ab/\sigma^2}$
- $B(t,T) = \frac{2(e^{(\gamma(T-t)}-1)}{(\gamma+a)(e^{\gamma(T-t)}-1)+2\gamma}$
- $\gamma = \sqrt{a^2 + 2\sigma^2}$

Example 1

- In script 6, we will fit the two models to the yield curve - CIR and Vasicek models.



Example 2

Present Value of liabilities, assets and surplus of our portfolio of 3 bonds, under yields approximated by CIR and Vasicek models:

```
  PV.liab.CIR PV.assets.CIR PV.surplus.CIR
1    11728816      11747649       18832.91
```

```
  PV.liab.vsk PV.assets.vsk PV.surplus.vsk
1    11721531      11740279       18748.45
```

- The surplus PV is different from 0 because the CIR model is `not a perfect replication of the empirical yield curve` that was used to find the immunising portfolio, and neither is the Vasicek model.

Simulation - CIR

- Discrete time approximation of the `CIR` model:

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dz(t)$$

- If Δt is small, this is approximated by

$$r(t + \Delta t) = r(t) + a(b - r(t))\Delta t + \sigma\sqrt{r(t)} \cdot \left[\sqrt{\Delta t} \cdot N(0,1)\right]$$

Simulation - Vasicek

- Discrete time approximation of the `Vasicek` model:

$$dr(t) = a(b - r(t))dt + \sigma' dz(t)$$

- If Δt is small, this is approximated by

$$r(t + \Delta t) = r(t) + a(b - r(t))\Delta t + \sigma' \cdot \left[\sqrt{\Delta t} \cdot N(0,1)\right]$$

- Note that $\sigma' \neq \sigma$ (from CIR).

Simulation - Central results

Simulation of a univariate random variable by `inverting the cumulative distribution` function:

- If $F^{-1}(u) = inf\{x : F(x) \geq u\}$ and $U \sim Unif[0, 1]$, then $F^{-1}(U) \sim F$.
- One can simulate any random variable $X \sim F$ by $X = F^{-1}\left(U\right)$, with $U \sim Unif\left[0, 1\right]$. In **R**, we may use the function `runif()`.
- If F has an explicit inverse or an available **R** function for the inverse, all becomes easy, e.g. `X = qnorm(runif())` will be normal distributed with mean `0` and s.d. `1`.

Example - I

- In `script 7`, we will compute 5000 six month year ahead simulations, using CIR and assuming all payments are made at year end. Homework: The same using Vasicek model.

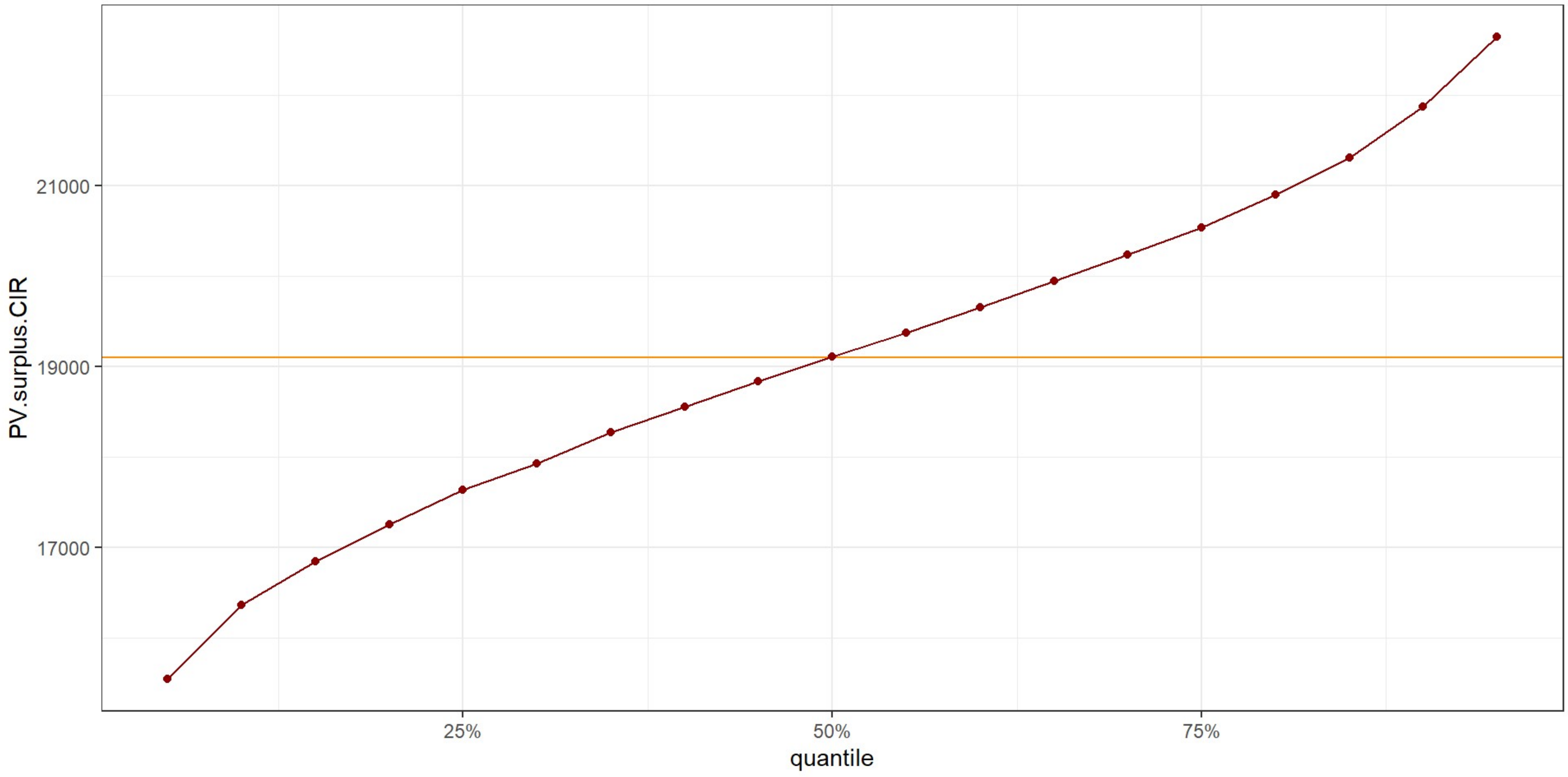
```
# A tibble: 75,000 × 9
     sim    r_0   drift   unif      dz rand.term  new.r `T-t` `P(t,T)`
   <int>  <dbl>   <dbl>  <dbl>   <dbl>     <dbl>  <dbl> <dbl>    <dbl>
 1     1 0.0195 0.00277 0.989   2.29    0.0113   0.0336   0.5    0.983
 2     2 0.0195 0.00277 0.398  -0.259 -0.00128  0.0210   0.5    0.989
 3     3 0.0195 0.00277 0.116  -1.20  -0.00591  0.0164   0.5    0.991
 4     4 0.0195 0.00277 0.0697 -1.48  -0.00730  0.0150   0.5    0.992
 5     5 0.0195 0.00277 0.244  -0.694 -0.00343  0.0188   0.5    0.990
 6     6 0.0195 0.00277 0.792   0.813  0.00402  0.0263   0.5    0.987
 7     7 0.0195 0.00277 0.340  -0.412 -0.00204  0.0202   0.5    0.989
 8     8 0.0195 0.00277 0.972   1.91   0.00944  0.0317   0.5    0.984
 9     9 0.0195 0.00277 0.166  -0.971 -0.00479  0.0175   0.5    0.991
10    10 0.0195 0.00277 0.459  -0.103 -0.000507 0.0218   0.5    0.989
# ℹ 74,990 more rows
```

Example - II

- First of the 5000 one-half year ahead simulations

```
# A tibble: 5,000 × 4
     sim PV.liab.CIR PV.assets.CIR PV.surplus.CIR
   <int>       <dbl>         <dbl>          <dbl>
 1     1   11495736.     11519726.         23990.
 2     2   11884984.     11903546.         18561.
 3     3   12032045.     12048573.         16528.
 4     4   12076503.     12092418.         15915.
 5     5   11952977.     11970597.         17620.
 6     6   11719234.     11740100.         20865.
 7     7   11908860.     11927090.         18231.
 8     8   11552177.     11575376.         23199.
 9     9   11996394.     12013414.         17020.
10    10   11860637.     11879536.         18899.
# ℹ 4,990 more rows
```

Example - Simulated PV of Surplus



Stochastic term structure models - no arbitrage

- The term structure of a Vasicek or CIR model can never perfectly replicate the actual (observed) term structure.
- Sometimes one needs a model that starts with `today's exact term structure` and evolves stochastically from there.
- Models with that property are called `no-arbitrage models`.

Ho-Lee and Hull-White models 1

- Like the Vasicek and CIR models, the `Ho-Lee` and `Hull-White` models have an affine term structure of the form

$$P(t,T) = A(t,T)e^{B(t,T)r(t)},$$

  with different functions A and B.
- In the Ho-Lee and Hull-White models, the function $A(t,T)$ depends not only on model parameters but also on the initial term structure.
- The Ho-Lee and Hull-White models permit perfect fit to today's empirical term structure.

Ho-Lee and Hull-White models 2

- Ho-Lee model

$$dr(t) = \theta(t)dt + \sigma dz(t)$$

- Hull-White model

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dz(t)$$

- The term $z(t)$ is standard Brownian motion (white noise).